How natural are law-invariant pricing rules?

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Valuation rules and law invariance

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A valuation rule is law invariant if prices depend only on the probability distribution of the corresponding payoffs:

law-invariant valuation rule
payoff's distribution $\Longrightarrow$ price

Two valuation approaches

How to value insurance liabilities?

$\Downarrow$

actuarial approach

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premium principles (law invariance is typically assumed)

How to value financial derivatives?

$\Downarrow$

math finance approach

$\Downarrow$

no-arbitrage principles (law invariance plays no role)

How to harmonize actuarial and financial valuation?

This question has been addressed in the literature and led to develop

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At a theoretical level one typically defines a market-consistent valuation rule by specifying a number of desirable (theoretical) properties.

In the presentation we focus on the question:

Is law invariance a desirable property of market-consistent valuation rules?

The modelling framework

The modelling framework

Throughout the talk we consider a simple one-period economy with dates

$$t = 0 \quad \text{and} \quad t = 1$$

We model uncertainty about the state of the economy at $t = 1$ by a

$$\text{probability space } (\Omega, \mathcal{F}, P)$$

The payoff of a financial contract at $t = 1$ is modelled as a

$$\text{random variable } X : \Omega \to \mathbb{R}$$

The set of all financial payoffs of interest is a vector space denoted by $\mathcal{X}$

$$\mathcal{X} = \{\text{financial payoffs}\} = \text{payoff space}$$

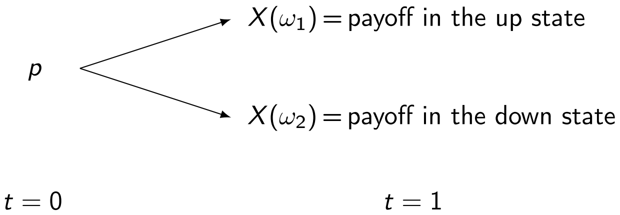
The modelling framework

In the simplest setting we have only two future states, ie

$$\Omega = \{\omega_1, \omega_2\} = \{\text{up}, \text{down}\}$$

$$p_1 = P(\omega_1), \quad p_2 = P(\omega_2)$$

In this setting a price-payoff couple can be visualized as



The actuarial approach

Actuarial approach: Premium principles

The set of all insurance payoffs is a convex set $\mathcal{C} \subset \mathcal{X}$, ie we set

$$\mathcal{C} = \{\text{insurance payoffs}\} = \text{insurance space}$$

Premia are determined according to a premium principle:

<u>premium principle</u>

$$\text{insurance payoff } X \quad \Longrightarrow \quad \text{premium } \pi_{act}(X)$$

In the classical insurance pricing literature premium principles are typically linked to utility functions.

Three typical properties of premium principles

- convexity, ie diversification leads to lower premia: for $\lambda \in [0,1]$

$$\pi_{act}(\lambda X + (1-\lambda)Y) \leq \lambda\pi_{act}(X) + (1-\lambda)\pi_{act}(Y)$$

- monotonicity, ie higher potential claims lead to higher premia:

$$Y \geq X \implies \pi_{act}(Y) \geq \pi_{act}(X)$$

- law invariance, ie premia depend only on the payoff distribution:

$$P_X = P_Y \implies \pi_{act}(X) = \pi_{act}(Y)$$

(where P_X and P_Y denote the probability law of X and Y).

Law invariance from which perspective?

Assume that an agent wants to insure his/her house against fire and

$$\text{probability that the house burns} = 50\%$$

Consider the contracts

5 → 100 if the house burns
  → 0 if the house does not burn

5 → 100 if heads turns up
  → 0 if tails turns up

Clearly, the insurer and the policyholder will look at the above contracts in a completely different way.

Law invariance is natural from the insurer's perspective (law of large numbers) but not from the policyholder's one.

The mathematical finance approach

Introducing the financial market

We assume that N financial securities are liquidly traded in the market and are characterized by the price-payoff couples

$$(S_0^1, S_1^1), \dots, (S_0^N, S_1^N)$$

The first security is a risk-free bond with interest rate $r \in (-1, \infty)$, ie

$$(S_0^1, S_1^1) = (1, 1 + r)$$

A portfolio of the N securities is modelled as a vector

$$w = (w^1, \dots, w^N)$$

Replicable payoffs

The price and payoff associated to a portfolio $w \in \mathbb{R}^N$ are given by

$$V_0(w) = \sum_{i=1}^{N} w^i S_0^i \quad \text{and} \quad V_1(w) = \sum_{i=1}^{N} w^i S_1^i$$

A payoff $X \in \mathcal{X}$ is replicable if there exists a portfolio $w \in \mathbb{R}^N$ such that

$$X = V_1(w)$$

The set of replicable payoffs is a vector subspace $\mathcal{M} \subset \mathcal{X}$, ie we set

$$\mathcal{M} = \{\text{replicable financial payoffs}\} = \text{marketed space}$$

Math finance approach: No-arbitrage principles

The market is arbitrage free holds if there is no $w \in \mathbb{R}^N$ such that

$$V_0(w) \leq 0, \quad V_1(w) \geq 0, \quad P(V_1(w) > 0) > 0.$$

Math finance approach: No-arbitrage principles

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Under no arbitrage, we can define a valuation rule as follows:

<u>no-arbitrage principle</u>

$$\text{replicable payoff } X = V_1(w) \implies \text{price } \pi_{na}(X) = V_0(w)$$

The quantity $\pi_{na}(X)$ can be interpreted as the replication price of X.

Math finance approach: No-arbitrage principles

Fundamental Theorem of Asset Pricing. Under no arbitrage, there exists a probability measure Q that is equivalent to P and satisfies

$$\pi_{na}(X) = \frac{E_Q(X)}{1+r}$$

for every replicable payoff $X \in \mathcal{M}$.

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for every replicable payoff $X \in \mathcal{M}$.

The measure Q allows to price replicable payoffs without having to first determine a replicating portfolio. For this reason, it is called a pricing measure.

Note that for every replicable payoff $X \in \mathcal{M}$ (with $\pi_{na}(X) \neq 0$) we have

$$E_Q\left(\frac{X - \pi_{na}(X)}{\pi_{na}(X)}\right) = r$$

This shows that, under Q, the expected rate of return on any replicable payoff coincides with the risk-free rate. For this reason, Q is also called a risk-neutral measure.

Market-consistent valuation rules

Market-consistent valuation rules

Definition. Let π_{act} be a premium principle. We say that a valuation rule

$$\text{payoff } X \quad \Longrightarrow \quad \text{price } \pi(X)$$

is a market-consistent extension of π_{act} if:

- $\pi(X) = \pi_{na}(X)$ for every replicable payoff $X \in \mathcal{M}$;
- $\pi(X) = \pi_{act}(X)$ for every insurance payoff $X \in \mathcal{C}$.

A market-consistent valuation rule can be interpreted as a generalized premium principle that prices replicable financial contracts in accordance to their replication cost (market consistent) and insurance contracts in accordance to the given premium principle (extension).

Market-consistent valuation rules

In 1995-2005 several authors tried to use premium principles such as

- proportional hazard tranforms (Wang's transforms)
- principles based on distorted probabilities
- principles induced by Choquet integrals

to price financial market risk. However, the resulting rules were not market consistent.

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The above valuation rules are all convex, monotone, and law invariant.

It can be easily shown that market consistency is compatible with both convexity and monotonicity.

Is market consistency compatible with law invariance?

The incompatibility between market consistency and law invariance

Result. Let π be a convex and monotone extension of a given premium principle π_{act}. The following are equivalent:

- π is market consistent and law invariant.
- For every payoff $X \in \mathcal{X}$ we have

$$\pi(X) = E_P\left(\frac{X}{1+r}\right)$$

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Expectations under P (the historical probability measure) are legitimate premium principles on the insurance space.

However, they become foolish valuation rules when applied to financial markets because they do not embed any premium for risk and, hence, are incompatible with risk-averse behaviours.

A special class of market-consistent valuation rules

A useful decomposition

Result. In the insurance literature it is customary to assume that

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In this case, every payoff $X \in \mathcal{X}$ can be uniquely decomposed as

$$X = X_{\mathcal{M}} + X_{\mathcal{C}}$$

for suitable payoffs $X_{\mathcal{M}} \in \mathcal{M}$ and $X_{\mathcal{C}} \in \mathcal{C}$

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for suitable payoffs $X_{\mathcal{M}} \in \mathcal{M}$ and $X_{\mathcal{C}} \in \mathcal{C}$ given by

$$X_{\mathcal{M}} = E_P(X|\text{market information}) \quad \text{and} \quad X_{\mathcal{C}} = X - X_{\mathcal{M}}$$

The interpretation is as follows:

- $X_{\mathcal{M}}$ is the hedgeable part of X
- $X_{\mathcal{C}}$ is the unhedgeable part of X

A special class of market-consistent valuation rules

Result. Let π_{act} be a given premium principle satifying

$$\pi_{act}(X + m) = \pi_{act}(X) + \frac{m}{1 + r} \quad \text{for every } m \in \mathbb{R}$$

The valuation rule defined by

$$\pi(X) = \pi_{na}(X_{\mathcal{M}}) + \pi_{act}(X_{\mathcal{C}})$$

is a market-consistent extension of π_{act}.

A special class of market-consistent valuation rules

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The valuation rule defined by

$$\pi(X) = \pi_{na}(X_{\mathcal{M}}) + \pi_{act}(X_{\mathcal{C}})$$

is a market-consistent extension of π_{act}. If π_{act} is law invariant, then π is law invariant on the subset $\mathcal{M}^{\perp}$.

A special class of market-consistent valuation rules

Result. Let ρ be a risk measure and define π_{act} by setting

$$\pi_{act}(X) = \frac{E_P(X)}{1+r} + \rho(X - E_P(X))$$

The valuation rule defined by

$$\pi(X) = \pi_{na}(X_{\mathcal{M}}) + \pi_{act}(X_{\mathcal{C}}) = \pi_{na}(X_{\mathcal{M}}) + \rho(X_{\mathcal{C}})$$

is a market-consistent extension of π_{act}.

This rule is consistent with the current regulatory valuation standards:

- $\frac{E_P(X)}{1+r}$ is the best estimate of the insurance claim X
- $\rho(X - E_P(X))$ is the risk margin for the insurance claim X

Conclusions

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  - finding a replicating portfolio for $E_P(X|\text{market information})$
  - modelling future scenarios (distributions are not enough)
- Computing $\rho(X_{\mathcal{C}})$ requires (provided ρ is law invariant)
  - building a statistical model for $X - E_P(X|\text{market information})$
  - selecting a good estimator for ρ

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- Computing $\rho(X_{\mathcal{C}})$ requires (provided ρ is law invariant)
  - building a statistical model for $X - E_P(X|\text{market information})$
  - selecting a good estimator for ρ
- The rule π_{na} is given by the gods of mathematics, but...
- How to choose ρ?

Thanks for your attention and enjoy the gala dinner!

Literature

- Bühlmann, H.: An economic premium principle, *ASTIN Bulletin*, 1980
- Bühlmann, H.: The general economic premium principle, *ASTIN Bulletin*, 1984
- Deprez, O., Gerber, H.U.: On convex principles of premium calculation, *Insurance: Mathematics and Economics*, 1985
- Delbaen, F., Haezendonck, J.: A martingale approach to premium calculation principles in an arbitrage free market, *Insurance: Mathematics and Economics*, 1989
- Venter, G.G.: Premium calculation implications of reinsurance without arbitrage, *ASTIN Bulletin*, 1991
- Albrecht, P.: Premium calculation without arbitrage? A note on a contribution by G. Venter, *ASTIN Bulletin*, 1992
- Bühlmann, H.: Stochastic discounting, *Insurance: Mathematics and Economics*, 1992
- Gerber, H.U., Shiu, E.S.W.: Option pricing by Esscher transforms, Transactions of the Society of Actuaries, 1994

Literature

- Chateauneuf, A., Kast, R., Lapied, A.: Choquet pricing for financial markets with frictions, *Mathematical Finance*, 1996
- Gerber, H.U., Shiu, E.S.W.: Actuarial bridges to dynamic hedging and option pricing, *Insurance: Mathematics ans Economics*, 1996
- Wang, S., Young, V., Panjer, H.: Axiomatic characterization of insurance prices, *Insurance: Mathematics and Economics*, 1997
- Bladt, M., Rydberg, T.H.: An actuarial approach to option pricing under the physical measure and without market assumptions, *Insurance: Mathematics ans Economics*, 1998
- Embrechts, P.: Actuarial versus financial pricing of insurance, *The Journal of Risk Finance*, 2000
- Wang, S.: A class of distortion operators for pricing financial and insurance risks, *Journal of Risk and Insurance*, 2000
- Schweizer, M.: From actuarial to financial valuation principles, *Insurance: Mathematics and Economics*, 2001
- Castagnoli, E., Maccheroni, F., Marinacci, M.: Insurance premia consistent with the market, *Insurance: Mathematics and Economics*, 2002

Literature

- Wang, S.: A universal framework for pricing financial and insurance risks, *ASTIN Bulletin*, 2002
- Castagnoli, E., Maccheroni, F., Marinacci, M.: Choquet insurance pricing: a caveat, *Mathematical Finance*, 2004
- Goovaerts, M.J., Laeven, R.J.A.: Actuarial risk measures for financial derivative pricing, *Insurance: Mathematics and Economics*, 2008
- Malamud, S., Trubowitz, E., Wüthrich, M.: Market consistent pricing of insurance products, *ASTIN Bulletin*, 2008
- Artzner, Ph., Eisele, K.T.: Supervisory insurance accounting: mathematics for provision – and solvency capital – requirements, *ASTIN Bulletin*, 2010
- Knispel, T., Stahl, G., Weber, S.: From the equivalence principle to market consistent valuation, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 2011
- Möhr, Ch.: Market-consistent valuation of insurance liabilities by cost of capital, *ASTIN Bulletin*, 2011
- Tsanakas, A., Wüthrich, M., Cerny, A.: Market value margin via mean-variance hedging, ASTIN Bulletin, 2013

Literature

- Bernard, C., Vanduffel, S.: Financial bounds for insurance claims, *Journal of Risk and Insurance*, 2014
- Natolski, J., Werner, R.: Mathematical analysis of different approaches for replicating portfolios, *European Actuarial Journal*, 2014
- Pelsser, A., Stadje, M.: Time-consistent and market-consistent evaluations, *Mathematical Finance*, 2014
- Dhaene, J., Stassen, B., Barigou, K., Linders, D., Chen, Z.: Fair valuation of insurance liabilities: Merging actuarial judgement and market-consistency, *Insurance: Mathematics and Economics*, 2017
- Engsner, H., Lindholm, M., Lindskog, F.: Insurance valuation: A computable multi-period cost-of-capital approach, *Insurance: Mathematics and Economics*, 2017
- Bellini, F., Koch-Medina, P., Munari, C., Svindland, G.: Law-invariant insurance pricing and its limitations, arXiv:1808.00821, 2018
- Cambou, M., Filipovic, D.: Replicating portfolio approach to capital calculation, *Finance and Stochastics*, 2018
- Natolski, J., Werner, R.: Mathematical foundations of the replicating portfolio approach, *Scandinavian Actuarial Journal*, 2018